

CONSTRUCTING KRALL-HAHN ORTHOGONAL POLYNOMIALS

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ABSTRACT. Given a sequence of polynomials $(p_n)_n$, an algebra of operators \mathcal{A} acting in the linear space of polynomials and an operator $D_p \in \mathcal{A}$ with $D_p(p_n) = \theta_n p_n$, where θ_n is any arbitrary eigenvalue, we construct a new sequence of polynomials $(q_n)_n$ by considering a linear combination of $m + 1$ consecutive p_n : $q_n = p_n + \sum_{j=1}^m \beta_{n,j} p_{n-j}$. Using the concept of \mathcal{D} -operator, we determine the structure of the sequences $\beta_{n,j}$, $j = 1, \dots, m$, in order that the polynomials $(q_n)_n$ are eigenfunctions of an operator in the algebra \mathcal{A} . As an application, from the classical discrete family of Hahn polynomials we construct orthogonal polynomials $(q_n)_n$ which are also eigenfunctions of higher-order difference operators.

1. INTRODUCTION

The issue of orthogonal polynomials which are also eigenfunctions of a higher-order differential operator was raised by H. L. Krall in 1939, when he obtained a complete classification for the case of a differential operator of order four ([23]). After his pioneer work, orthogonal polynomials which are also eigenfunctions of higher-order differential operators are usually called Krall polynomials. This terminology can be extended for finite order difference and q -difference operators. Krall polynomials are also called bispectral, following the terminology introduced by Duistermaat and Grünbaum ([6]; see also [13], [14]).

Regarding Krall polynomials, there are important differences depending whether one considers differential or difference operators. Indeed, roughly speaking, one can construct Krall polynomials $q_n(x)$, $n \geq 0$, by using the Laguerre $x^\alpha e^{-x}$, or Jacobi weights $(1-x)^\alpha(1+x)^\beta$, assuming that one or two of the parameters α and β are nonnegative integers and adding a linear combination of Dirac deltas and their derivatives at the endpoints of the orthogonality interval ([23], [20], [21], [22], [24], [25], [13], [15], [16], [18], [19], [27]). This procedure of adding deltas seems not to work if we want to construct Krall discrete polynomials from the classical discrete measures of Charlier, Meixner, Krawtchouk and Hahn (see the papers [2] and [3] by Bavinck, van Haeringen and Koekoek answering, in the negative, a question posed by R. Askey in 1991 (see pag. 418 of [4])).

As it has been shown recently by one of us, instead of adding deltas, Krall discrete polynomials can be constructed by multiplying the classical discrete weights by certain polynomials (see [8]). The kind of transformation which consists in

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multiplying a measure μ by a polynomial r is called a Christoffel transform. It has a long tradition in the context of orthogonal polynomials: it goes back a century and a half ago when E. B. Christoffel (see [5] and also [26]) studied it for the particular case $r(x) = x$.

Using suitable polynomials, a number of conjectures have been posed in [8] on how to construct bispectral polynomials from the families of Charlier, Meixner, Krawtchouk and Hahn. For Charlier, Meixner and Krawtchouk, those conjectures have been proved by the authors in [11]. Those families have in common that the eigenvalues for their associated second-order difference operator are linear sequences in n .

The purpose of this paper is to prove the conjecture 5 in [8] for the Hahn polynomials. In this case, the eigenvalues for its associated second-order difference operator are now a second degree polynomial in n . We consider higher-order difference operators of the form

$$(1.1) \quad D = \sum_{l=s}^r h_l \mathfrak{s}_l, \quad s \leq r, \quad s, r \in \mathbb{Z},$$

where h_l are polynomials and \mathfrak{s}_l stands for the shift operator $\mathfrak{s}_l(p) = p(x+l)$. If $h_r, h_s \neq 0$, the order of D is then $r-s$. We also say that D has genre (s, r) .

The conjecture we will prove here is the following:

Conjecture. Let $\rho_{a,b,N}$ be the Hahn weight (see Section 5 for details). Given a quartet of finite sets $\mathcal{F} = (F_1, F_2, F_3, F_4)$ of positive integers (the empty set is allowed) consider the weight $\rho_{a,b,N}^{\mathcal{F}}$ defined by

$$(1.2) \quad \rho_{a,b,N}^{\mathcal{F}} = \prod_{f \in F_1} (b+N+1+f-x) \prod_{f \in F_2} (x+a+1+f) \prod_{f \in F_3} (N-f-x) \prod_{f \in F_4} (x-f) \rho_{a,b,N}.$$

Assume that the measure $\rho_{a,b,N}^{\mathcal{F}}$ has an associated sequence of orthogonal polynomials. Then they are eigenfunctions of a higher-order difference operator of the form (1.1) with

$$-s = r = \sum_{f \in F_1, F_2, F_3, F_4} f - \sum_{i=1}^4 \binom{n_{F_i}}{2} + 1,$$

where n_F denotes the cardinal of F .

The content of this paper is as follows. In order to prove the conjecture above, we use the approach developed in [11] for constructing Krall polynomials from the Charlier, Meixner and Krawtchouk families. The main ingredients of this approach will be considered in Sections 3 and 4. The first ingredient is the \mathcal{D} -operators (see Section 3). This is an abstract concept introduced in [9] by one of us which has shown to be very useful to generate Krall, Krall discrete and q -Krall families of polynomials (see [9], [1], [11], [12], [10]). For a positive number m and m polynomials Y_j , $j = 1, \dots, m$, (which act as parameters) we can construct from the classical discrete families $(p_n)_n$, using \mathcal{D} -operators, a huge class of families of polynomials $(q_n)_n$ which are also eigenfunctions of difference operators of the form (1.1). The sequence of polynomials $(q_n)_n$ are not in general orthogonal. The second ingredient establishes how to choose the polynomials Y_j 's such that the polynomials $(q_n)_n$ are also orthogonal with respect to a measure. As for the case of Charlier, Meixner and

Krawtchouk (studied in [11]), this second ingredient turns into a very nice symmetry between the Hahn family and the polynomials Y_j 's. Indeed, the polynomials Y_j 's can be chosen to be dual Hahn polynomials, but with a suitable modification of the parameters.

In Sections 5 and 6 we will put together all these ingredients to construct bispectral Krall-Hahn orthogonal polynomials and prove the conjecture above.

2. PRELIMINARIES

For a linear operator D acting in the linear space of polynomials \mathbb{P} , i.e. $D : \mathbb{P} \rightarrow \mathbb{P}$, and a polynomial $P(x) = \sum_{j=0}^k a_j x^j$, the operator $P(D)$ is defined in the usual way as $P(D) = \sum_{j=0}^k a_j D^j$.

Let μ be a moment functional on the real line, that is, a linear mapping $\mu : \mathbb{P} \rightarrow \mathbb{R}$. It is well-known that any moment functional on the real line can be represented by integrating with respect to a Borel measure (positive or not) on the real line (this representation is not unique, see [7]). If we also denote this measure by μ , we have $\langle \mu, p \rangle = \int p(x) d\mu(x)$ for all polynomials $p \in \mathbb{P}$. Taking this into account, we will conveniently use along this paper one or other terminology (orthogonality with respect to a moment functional or with respect to a measure). We say that a sequence of polynomials $(p_n)_n$, p_n of degree n , $n \geq 0$, is orthogonal with respect to the moment functional μ if $\langle \mu, p_n p_m \rangle = 0$, for $n \neq m$ and $\langle \mu, p_n^2 \rangle \neq 0$. Since the Hahn polynomials considered in this paper are orthogonal with respect to a degenerate measure (the measure is a finite combination of Dirac deltas), we will stress this property of non-vanishing norms when necessary.

As we wrote in the Introduction, the kind of transformation which consists in multiplying a moment functional μ by a polynomial r is called a Christoffel transform. The new moment functional $r\mu$ is defined by $\langle r\mu, p \rangle = \langle \mu, rp \rangle$. Its reciprocal is the Geronimus transform $\tilde{\mu}$ which satisfies $r\tilde{\mu} = \mu$. Notice that the Geronimus transform of the moment functional μ is not uniquely defined. Indeed, write a_i , $i = 1, \dots, u$, for the different real roots of the polynomial r , each one with multiplicity b_i , respectively. It is easy to see that if $\tilde{\mu}$ is a Geronimus transform of μ then the moment functional $\tilde{\mu} + \sum_{i=1}^u \sum_{j=0}^{b_i-1} \alpha_{i,j} \delta_{a_i}^{(j)}$ is also a Geronimus transform of μ , where $\alpha_{i,j}$ are real numbers. These numbers are usually called the free parameters of the Geronimus transform.

In the literature, Geronimus transform is sometimes called Darboux transform with parameters while Christoffel transform is called Darboux transform without parameters. The reason is the following. The three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + c_np_{n-1}(x), \quad n \geq 0,$$

for the orthogonal polynomials $(p_n)_n$ with respect to μ can be rewritten as $xp_n = J(p_n)$, where J is the second-order difference operator $J = a_{n+1}\mathfrak{s}_1 + b_n\mathfrak{s}_0 + c_n\mathfrak{s}_{-1}$ and \mathfrak{s}_l is the shift operator (acting on the discrete variable n): $\mathfrak{s}_l(x_n) = x_{n+l}$. For any $\lambda \in \mathbb{C}$, decompose J into $J = AB + \lambda I$ whenever it is possible, where $A = \alpha_n\mathfrak{s}_0 + \beta_n\mathfrak{s}_1$ and $B = \delta_n\mathfrak{s}_{-1} + \gamma_n\mathfrak{s}_0$. We then call $\tilde{J} = BA + \lambda I$ a Darboux transform of J with parameter λ . It turns out that the second-order difference operator \tilde{J} associated to a Geronimus transform $\tilde{\mu}$ of μ can be obtained by applying a sequence of k Darboux transforms (with parameters λ_i , $i = 1, \dots, k$) to the operator J associated to the measure μ . This kind of Darboux transform has been

used by Grünbaum, Haine, Hozorov, Yakimov and Iliev to construct Krall and q -Krall polynomials ([15], [16], [17], [18] or [19]).

The family of measures $\rho_{a,b,N}^{\mathcal{F}}$ (1.2) in the Introduction is defined by applying a Christoffel transform to the Hahn weight. But, it turns out that they can also be defined by using the Geronimus transform. This Geronimus transform is however defined by a different polynomial. We also have to make a suitable choice of the free parameters of this Geronimus transform and apply it to a Hahn weight but maybe with different parameters and affected by a shift in the variable. The following example will clarify this point. Consider $F_1 = \{1\}$, $F_2 = \{1\}$, $F_3 = \{1\}$, $F_4 = \{1\}$ and the Christoffel transform $\rho_{a,b,N}^{\mathcal{F}}$ of the Hahn weight $\rho_{a,b,N}$ defined by the polynomial $p(x) = (b + N + 2 - x)(x + a + 2)(N - 1 - x)(x - 1)$. That is,

$$\rho_{a,b,N}^{\mathcal{F}} = (b + N + 2 - x)(x + a + 2)(N - 1 - x)(x - 1)\rho_{a,b,N}.$$

From the definition of the Hahn weight $\rho_{a,b,N}$ (see (5.5) below), we have after a simple computation

$$\begin{aligned} \rho_{a,b,N}^{\mathcal{F}} = & -\frac{\Gamma(N + b + 3)(N - 1)(a + 2)}{(N + b + 1)\Gamma(b + 1)}\delta_0 - \frac{\Gamma(N + a + 3)(N - 1)(b + 2)}{(N + a + 1)\Gamma(a + 1)}\delta_N \\ & - \frac{\Gamma(N + 1)}{\Gamma(a + 1)\Gamma(b + 1)} \sum_{x=2}^{N-2} \frac{\Gamma(N - x + b + 3)\Gamma(x + a + 3)}{(a + x + 1)(N - x + b + 1)x(x - 2)!(N - x - 2)!}\delta_x \end{aligned}$$

This shows that

$$x(N - x)(a + x + 1)(N - x + b + 1)\rho_{a,b,N}^{\mathcal{F}} = (N - 3)_4(a + 1)_4(b + 1)_4\rho_{a+4,b+4,N-4}(x - 2),$$

where $(z)_j = z(z + 1) \cdots (z + j - 1)$ stands for the Pochhammer symbol.

That is, $\rho_{a,b,N}^{\mathcal{F}}$ is also the Geronimus transform defined by the polynomial $x(N - x)(a + x + 1)(N - x + b + 1)$ of the Hahn weight $\rho_{a+4,b+4,N-4}(x - 2)$ where the free parameters (associated to the roots $x = 0, N, -a - 1, N + b + 1$) have to be necessarily chosen equal to $-\frac{\Gamma(N + b + 3)(N - 1)(a + 2)}{(N + b + 1)\Gamma(b + 1)}$, $\frac{\Gamma(N + a + 3)(N - 1)(b + 2)}{(N + a + 1)\Gamma(a + 1)}$, 0 and 0, respectively.

Along this paper, we use the following notation: given a finite set of positive integers $F = \{f_1, \dots, f_m\}$, the expression

$$(2.1) \quad \left[\begin{array}{c} z_{f,j} \\ f \in F \end{array} \right]^{j=1,\dots,m}$$

inside of a matrix or a determinant will mean the submatrix defined by

$$\begin{pmatrix} z_{f_1,1} & z_{f_1,2} & \cdots & z_{f_1,m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{f_m,1} & z_{f_m,2} & \cdots & z_{f_m,m} \end{pmatrix}$$

3. \mathcal{D} -OPERATORS

The concept of \mathcal{D} -operator was introduced by one of us in [9]. In [9], [11], [12] and [1], it has been showed that \mathcal{D} -operators turn out to be an extremely useful tool of an unified method to generate families of polynomials which are eigenfunctions

of higher-order differential, difference or q -difference operators. Hence, we start by reminding the concept of \mathcal{D} -operator.

The starting point is a sequence of polynomials $(p_n)_n$, $\deg p_n = n$, and an algebra of operators \mathcal{A} acting in the linear space of polynomials \mathbb{P} . For the Hahn polynomials, we will consider the algebra \mathcal{A} formed by all finite order difference operators, i.e.

$$(3.1) \quad \mathcal{A} = \left\{ \sum_{l=s}^r h_l \mathfrak{s}_l : h_l \in \mathbb{P}, l = s, \dots, r, s \leq r \right\},$$

where \mathfrak{s}_l stands for the shift operator $\mathfrak{s}_l(p) = p(x+l)$. If $h_r, h_s \neq 0$, the order of D is then $r-s$. We also say that D has genre (s, r) .

In addition, we assume that the polynomials p_n , $n \geq 0$, are eigenfunctions of certain operator $D_p \in \mathcal{A}$. We write $(\theta_n)_n$ for the corresponding eigenvalues, so that $D_p(p_n) = \theta_n p_n$, $n \geq 0$. Although for the Hahn polynomials θ_n is a second degree polynomial in n , in this section we do not assume any constraint on the sequence $(\theta_n)_n$.

Given two sequences of numbers $(\varepsilon_n)_n$ and $(\sigma_n)_n$, a \mathcal{D} -operator associated to the algebra \mathcal{A} and the sequence of polynomials $(p_n)_n$ is defined as follows. We first consider the operator $\mathcal{D} : \mathbb{P} \rightarrow \mathbb{P}$ defined by linearity from

$$\mathcal{D}(p_n) = -\frac{1}{2}\sigma_{n+1}p_n + \sum_{j=1}^n (-1)^{j+1} \sigma_{n-j+1} \varepsilon_n \cdots \varepsilon_{n-j+1} p_{n-j}, \quad n \geq 0.$$

We then say that \mathcal{D} is a \mathcal{D} -operator if $\mathcal{D} \in \mathcal{A}$. In [9] this kind of \mathcal{D} -operator was called of type 2; \mathcal{D} -operators of type 1 appears when the sequence $(\sigma_n)_n$ is constant.

As it was shown in [9], [10] and [11], \mathcal{D} -operators of type 1 are useful when the eigenvalues θ_n are linear in n . Otherwise, we have to consider \mathcal{D} -operators of type 2. The purpose of this section is to extend the method developed in [11] to \mathcal{D} -operators of type 2. Therefore we will be able to construct, from the sequence $(p_n)_n$, new sequences of polynomials $(q_n)_n$ such that there exists an operator $D_q \in \mathcal{A}$ for which they are eigenfunctions.

To do that, we will consider a combination of m , $m \geq 1$, consecutive p_n 's. We also use m arbitrary polynomials Y_1, Y_2, \dots, Y_m , and m \mathcal{D} -operators, $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$, (not necessarily different) defined by the pairs of sequences $(\varepsilon_n^h)_n$, $(\sigma_n^h)_n$, $h = 1, \dots, m$:

$$(3.2) \quad \mathcal{D}_h(p_n) = -\frac{1}{2}\sigma_{n+1}^h p_n + \sum_{j=1}^n (-1)^{j+1} \sigma_{n-j+1}^h \varepsilon_n^h \cdots \varepsilon_{n-j+1}^h p_{n-j}.$$

We will assume that for $h = 1, 2, \dots, m$, the sequences $(\varepsilon_n^h)_n$ and $(\sigma_n^h)_n$ are rational functions in n . We write $\xi_{x,i}^h$, $i \in \mathbb{Z}$ and $h = 1, 2, \dots, m$, for the auxiliary functions defined by

$$(3.3) \quad \xi_{x,i}^h = \prod_{j=0}^{i-1} \varepsilon_{x-j}^h, \quad i \geq 1, \quad \xi_{x,0}^h = 1, \quad \xi_{x,i}^h = \frac{1}{\xi_{x-i,-i}^h}, \quad i \leq -1.$$

We will consider the $m \times m$ (quasi) Casorati determinant defined by

$$(3.4) \quad \Omega(x) = \det \left(\xi_{x-j, m-j}^l Y_l(\theta_{x-j}) \right)_{l,j=1}^m.$$

The details of our method are included in the following Theorem:

Theorem 3.1. *Let \mathcal{A} and $(p_n)_n$ be, respectively, an algebra of operators acting in the linear space of polynomials, and a sequence of polynomials $(p_n)_n$, $\deg p_n = n$. We assume that $(p_n)_n$ are eigenfunctions of an operator $D_p \in \mathcal{A}$, that is, there exist numbers $\theta_n, n \geq 0$, such that $D_p(p_n) = \theta_n p_n$, $n \geq 0$. We also have m pairs of sequences of numbers $(\varepsilon_n^h)_n, (\sigma_n^h)_n$, $h = 1, \dots, m$, which define m \mathcal{D} -operators $\mathcal{D}_1, \dots, \mathcal{D}_m$ (not necessarily different) for $(p_n)_n$ and \mathcal{A} (see (3.2)) and assume that for $h = 1, 2, \dots, m$, each one of the sequences $(\varepsilon_n^h)_n, (\sigma_n^h)_n$ is a rational function in n .*

Let Y_1, Y_2, \dots, Y_m , be m arbitrary polynomials satisfying that $\Omega(n) \neq 0$, $n \geq 0$, where Ω is the Casorati determinant defined by (3.4).

Consider the sequence of polynomials $(q_n)_n$ defined by

$$(3.5) \quad q_n(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi_{n,m}^1 Y_1(\theta_n) & \xi_{n-1,m-1}^1 Y_1(\theta_{n-1}) & \cdots & Y_1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m Y_m(\theta_n) & \xi_{n-1,m-1}^m Y_m(\theta_{n-1}) & \cdots & Y_m(\theta_{n-m}) \end{vmatrix}.$$

For a rational function S , we define the function λ_x by

$$(3.6) \quad \lambda_x - \lambda_{x-1} = S(x)\Omega(x),$$

and for $h = 1, \dots, m$, we define the function $M_h(x)$ by

$$(3.7) \quad M_h(x) = \sum_{j=1}^m (-1)^{h+j} \xi_{x,m-j}^h S(x+j) \det \left(\xi_{x+j-r,m-r}^l Y_l(\theta_{x+j-r}) \right)_{l \in \mathbb{I}_h; r \in \mathbb{I}_j},$$

where $\mathbb{I}_h = \{1, 2, \dots, m\} \setminus \{h\}$. Assume the following:

$$(3.8) \quad S(x)\Omega(x) \text{ is a polynomial in } x.$$

$$(3.9) \quad \text{There exist } \tilde{M}_1, \dots, \tilde{M}_m, \text{ polynomials in } x \text{ such that}$$

$$M_h(x) = \sigma_{x+1}^h \tilde{M}_h(\theta_x), \quad h = 1, \dots, m.$$

$$(3.10) \quad \text{There exists a polynomial } P_S \text{ such that } P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^m Y_h(\theta_x) M_h(x).$$

Then there exists an operator $D_{q,S} \in \mathcal{A}$ such that

$$D_{q,S}(q_n) = \lambda_n q_n, \quad n \geq 0.$$

Moreover, the operator $D_{q,S}$ is defined by

$$(3.11) \quad D_{q,S} = \frac{1}{2} P_S(D_p) + \sum_{h=1}^m \tilde{M}_h(D_p) \mathcal{D}_h Y_h(D_p),$$

where $D_p \in \mathcal{A}$ is the operator for which the polynomials $(p_n)_n$ are eigenfunctions.

Proof. The definition of (3.2), (3.6) and (3.11), along with the hypotheses (3.8)-(3.10), gives

$$\begin{aligned} D_{q,S}(p_n) &= \left(\frac{1}{2} P_S(\theta_n) - \frac{1}{2} \sum_{h=1}^m \sigma_{n+1}^h \tilde{M}_h(\theta_n) Y_h(\theta_n) \right) p_n \\ &\quad + \sum_{h=1}^m \sum_{j=1}^n (-1)^{j+1} \tilde{M}_h(\theta_{n-j}) \varepsilon_n^h \cdots \varepsilon_{n-j+1}^h \sigma_{n-j+1}^h Y_h(\theta_n) p_{n-j} \\ &= \lambda_n p_n + \lambda_{n,1} p_{n-1} - \lambda_{n,2} p_{n-2} + \cdots + (-1)^{m+1} \lambda_{n,m} p_{n-m} \\ &\quad + \sum_{j=m+1}^n (-1)^{j+1} \lambda_{n,j} p_{n-j} \end{aligned}$$

where

$$\lambda_{n,j} = \sum_{h=1}^m \varepsilon_n^h \cdots \varepsilon_{n-j+1}^h M_h(n-j) Y_h(\theta_n), \quad j = 1, 2, \dots, n.$$

Now we just have to follow exactly the same lines as the proof in Section 8 of [11] (from the formula (8.4) on). \square

Remark 3.2. We now see that the polynomial P_S (3.10) also satisfies

$$(3.12) \quad P_S(\theta_x) - P_S(\theta_{x-1}) = S(x)\Omega(x) + S(x+m)\Omega(x+m).$$

Indeed, P_S defined in (3.10) can be written, using (8.4) and (8.6) in the proof of Theorem 3.2 of [11], as

$$P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^m S(x+h)\Omega_{-h+1,h}(x+1),$$

where $\Omega_{i,j}(x)$ is the $m \times m$ determinant defined in the same way as $\Omega(x-i)$ but replacing the j -th column of $\Omega(x-i)$ by the column vector

$$\left(\xi_{x-1,m+i-1}^1 Y_1(\theta_{x-1}), \dots, \xi_{x-1,m+i-1}^m Y_m(\theta_{x-1}) \right)^t.$$

In particular $\Omega_{0,1}(x) = \Omega(x)$.

Now, using (3.6) and the relations (which easily follows by definition)

$$\Omega_{-h+1,h}(x+1) = \Omega_{-h,h+1}(x), \quad h = 1, \dots, m-1,$$

and

$$\Omega_{-m+1,m}(x+1) = \Omega_{0,1}(x+m) = \Omega(x+m),$$

we obtain (3.12).

Remark 3.3. For the particular cases of Laguerre, Jacobi or Askey-Wilson polynomials, one can found Casorati determinants similar to (3.5) in [15], [16], [17], [18] or [19].

In Sections 5 and 6, we will apply Theorem 3.1 to the Hahn polynomials. We will see there that the degree of the polynomial P_S (see (3.10)) gives the order of the difference operator $D_{q,S}$ (3.11) with respect to which the new polynomials $(q_n)_n$ are eigenfunctions. This will be a consequence of the following Lemma:

Lemma 3.4. *With the same notation as in Theorem 3.1, write*

$$\Psi_j^h(x) = \xi_{x-j, m-j}^h S(x) \det (\xi_{x-r, m-r}^l Y_l(\theta_{x-r}))_{l \in \mathbb{I}_h; r \in \mathbb{I}_j}, \quad h, j = 1, \dots, m,$$

and Ω_g^h , $h = 1, \dots, m$, $g = 0, 1, 2, \dots$, for the particular case of Ω when $Y_h(x) = x^g$. Assume that θ_x is a polynomial in x of degree 2 and that Ψ_j^h , $h, j = 1, \dots, m$, are polynomials in x and write $\tilde{d} = \max\{\deg \Psi_j^h : h, j = 1, \dots, m\}$. Then M_h and $S\Omega_g^h$, $h = 1, \dots, m$, $g = 0, 1, 2, \dots$, are also polynomials in x . If, in addition, we assume that the degree of $S\Omega_g^h$ is at most $2g + \deg(S\Omega_0^h)$ for $h = 1, \dots, m$, and $g \leq \tilde{d} - \deg(S\Omega_0^h)$, then M_h is a polynomial of degree at most $\deg(S\Omega_0^h)$.

Proof. It is analogous to the proof of Lemma 3.2 in [10]. \square

4. TWO MORE INGREDIENTS

In this Section we will assume that the polynomials $(p_n)_n$, p_n of degree n , $n \geq 0$, are orthogonal with respect to a moment functional ρ . We automatically have that the polynomials $(p_n)_n$ satisfy the three-term recurrence relation ($p_{-1} = 0$)

$$(4.1) \quad xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \geq 0.$$

The measure ρ might be degenerate, in which case for some n_0 we might have $a_{n_0}c_{n_0} = 0$.

The goal of this Section is to show how to choose appropriately the polynomials Y_1, \dots, Y_m , in order that the polynomials $(q_n)_n$ (3.5) are also orthogonal with respect to certain measure. To stress the dependence of the polynomials $(q_n)_n$ on the polynomials Y_1, \dots, Y_m , we write $q_n = \text{Cas}_n^{Y_1, \dots, Y_m}$.

4.1. When are the polynomials $(\text{Cas}_n^{Y_1, \dots, Y_m})_n$ orthogonal? Only for a convenient choice of the polynomials Y_j , $j = 1, \dots, m$, the polynomials $(q_n)_n$ (3.5) are also orthogonal with respect to a measure. In Section 4 of [11], a method to check the orthogonality of the polynomials $(q_n)_n$ (3.5) was provided. This tool assumed that the sequences $(\varepsilon_n^h)_{n \in \mathbb{Z}}$, $h = 1, \dots, m$, do not vanish for any n . This is not the case for the Hahn polynomials, hence we use the modified version of that tool given in [10] which includes also that case.

Indeed, given the sequences (in the integers) of the three-term recurrence relation $(a_n)_{n \in \mathbb{Z}}$, $(b_n)_{n \in \mathbb{Z}}$, $(c_n)_{n \in \mathbb{Z}}$ (see (4.1)) and $(\varepsilon_n^h)_{n \in \mathbb{Z}}$, $h = 1, \dots, m$, assume we have for each $j \geq 0$ and $h = 1, \dots, m$, one more sequence denoted by $(Z_j^h(n))_{n \in \mathbb{Z}}$ satisfying

$$(4.2) \quad \varepsilon_{n+1}^h a_{n+1} Z_j^h(n+1) - b_n Z_j^h(n) + \frac{c_n}{\varepsilon_n^h} Z_j^h(n-1) = (\eta_h j + \kappa_h) Z_j^h(n), \quad n \in \mathbb{Z},$$

where, η_h and κ_h are real numbers independent of n and j , and we assume that if for some n_0 and h_0 , $\varepsilon_{n_0}^{h_0} = 0$, then also $c_{n_0} = 0$ and there is a number $d_{n_0}^{h_0}$ such that the identity (4.2) still holds when we replace $c_{n_0}/\varepsilon_{n_0}^{h_0}$ by $d_{n_0}^{h_0}$.

Define the auxiliary numbers $\xi_{n,i}^h$, $i \geq 0$, $n \in \mathbb{Z}$ and $h = 1, \dots, m$, by

$$\xi_{n,i}^h = \prod_{j=n-i+1}^n \varepsilon_j^h, \quad i \geq 1, \quad \xi_{n,0}^h = 1, \quad \xi_{n,i}^h = \frac{1}{\xi_{n-i,-i}^h}, \quad i \leq -1.$$

For $i < 0$, we take $\xi_{n,i}^h = \infty$ if $\xi_{n-i,-i}^h = 0$ (this can happen if for some n_0, h_0 , $\varepsilon_{n_0}^{h_0} = 0$). However, by assuming $\varepsilon_n^h \neq 0$, $n \leq 0$, $h = 1, \dots, m$, one can straightforwardly

check that $\xi_{n,n+1}^h$ is finite for $n \leq 0$. Notice that for $x = n$, the number $\xi_{n,i}^h$ coincides with the number defined by (3.3).

Given a m -tuple G of m positive integers, $G = (g_1, \dots, g_m)$, assume that the m numbers

$$(4.3) \quad \tilde{g}_h = (\eta_h g_h + \kappa_h), \quad h = 1, \dots, m,$$

are different. Call $\tilde{G} = \{\tilde{g}_1, \dots, \tilde{g}_m\}$. Consider finally the $m \times m$ Casorati determinant Ω_G defined by

$$\Omega_G(x) = \det \left(\xi_{x,m-j}^l Z_{g_l}^l(\theta_{x-j}) \right)_{l,j=1}^m.$$

We then define the sequence of polynomials $(q_n^G)_n$ by

$$(4.4) \quad q_n^G(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi_{n,m}^1 Z_{g_1}^1(\theta_n) & \xi_{n-1,m-1}^1 Z_{g_1}^1(\theta_{n-1}) & \cdots & Z_{g_1}^1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m Z_{g_m}^m(\theta_n) & \xi_{n-1,m-1}^m Z_{g_m}^m(\theta_{n-1}) & \cdots & Z_{g_m}^m(\theta_{n-m}) \end{vmatrix}.$$

Notice that if for each $h = 1, \dots, m$, $Y_h(x) = Z_{g_h}^h(x)$ is a polynomial in x , satisfying the hypotheses of Theorem 3.1, then the polynomials q_n^G (4.4) fit into the definition of the polynomials (3.5) in Theorem 3.1, and hence they are eigenfunctions of an operator in the algebra \mathcal{A} .

The key to prove that the polynomials $(q_n^G)_n$ are orthogonal with respect to a measure $\tilde{\rho}$ are the following formulas. Assume that $\varepsilon_n^h \neq 0$, $n \leq 0$, $h = 1, \dots, m$, and that there exists a constant $c_G \neq 0$ such that

$$(4.5) \quad \langle \tilde{\rho}, p_n \rangle = (-1)^n c_G \sum_{i=1}^m \frac{\xi_{n,n+1}^i Z_{g_i}^i(\theta_n)}{\mathbf{p}'_{\tilde{G}}(\tilde{g}_i) Z_{g_i}^i(\theta_{-1})}, \quad n \geq 0,$$

$$(4.6) \quad 0 = \sum_{i=1}^m \frac{Z_{g_i}^i(\theta_n)}{\mathbf{p}'_{\tilde{G}}(\tilde{g}_i) \xi_{-1,-n-1}^i Z_{g_i}^i(\theta_{-1})}, \quad 1-m \leq n < 0,$$

$$(4.7) \quad 0 \neq \sum_{i=1}^m \frac{Z_{g_i}^i(\theta_{-m})}{\mathbf{p}'_{\tilde{G}}(\tilde{g}_i) \xi_{-1,m-1}^i Z_{g_i}^i(\theta_{-1})},$$

where \tilde{g}_h are the m different numbers (4.3) and $\mathbf{p}_{\tilde{G}}(x) = \prod_{i=1}^m (x - \tilde{g}_i)$.

We then have the following version of Lemma 4.2 of [11]:

Lemma 4.1 (Lemma 3.4 of [10]). *Assume that $\varepsilon_n^h \neq 0$, $n \leq 0$, $h = 1, \dots, m$, and that there exist M, N (each of them can be either a positive integer or infinity) such that $a_n c_n \neq 0$ for $1 \leq n \leq N$ and $\Omega_G(n) \neq 0$ for $0 \leq n \leq M$. Assume also that (4.5), (4.6) and (4.7) hold. Then the polynomials q_n^G , $0 \leq n \leq \min\{M-1, N+m\}$, are orthogonal with respect to $\tilde{\rho}$ and have non-null norms.*

4.2. Finite sets of positive integers. We still need a last ingredient for identifying the measure $\tilde{\rho}$ with respect to which the polynomials $(q_n^G)_n$ (4.4) are orthogonal. The measures $\rho_{a,b,N}^{\mathcal{F}}$ (1.2) in the Introduction depends on certain finite sets F_1, F_2, F_3 and F_4 while the polynomials $(q_n^G)_n$ depend on the finite set G (the degrees of the polynomials Z 's). The relationship between the sets F 's and G will be given by the following transforms of finite sets of positive integers.

Consider the sets Υ and Υ_0 formed by all finite sets of positive or nonnegative integers, respectively:

$$\begin{aligned}\Upsilon &= \{F : F \text{ is a finite set of positive integers}\}, \\ \Upsilon_0 &= \{F : F \text{ is a finite set of nonnegative integers}\}.\end{aligned}$$

We consider an involution I in Υ , and a family J_h , $h \geq 1$, of transforms from Υ into Υ_0 . For $F \in \Upsilon$ write $F = \{f_1, \dots, f_k\}$ with $f_i < f_{i+1}$, so that $f_k = \max F$. Then $I(F)$ and $J_h(F)$, $h \geq 1$, are defined by

$$\begin{aligned}(4.8) \quad I(F) &= \{1, 2, \dots, f_k\} \setminus \{f_k - f, f \in F\}, \\ (4.9) \quad J_h(F) &= \{0, 1, 2, \dots, f_k + h - 1\} \setminus \{f - 1, f \in F\}.\end{aligned}$$

For the involution I , the bigger the holes in F (with respect to the set $\{1, 2, \dots, f_k\}$), the bigger the involuted set $I(F)$. Here it is a couple of examples

$$I(\{1, 2, 3, \dots, k\}) = \{k\}, \quad I(\{1, k\}) = \{1, 2, \dots, k - 2, k\}.$$

Something similar happens for the transform J_h with respect to $\{0, 1, \dots, f_k + h - 1\}$.

Notice that

$$\max F = \max I(F), \quad h - 1 + \max F = \max J_h(F),$$

and if n_F denotes the cardinal of F , we also have

$$(4.10) \quad n_{I(F)} = f_k - n_F + 1, \quad n_{J_h(F)} = f_k + h - n_F.$$

For a quartet $\mathcal{F} = (F_1, F_2, F_3, F_4)$ of finite sets of positive integers, we will write $F_i = \{f_1^{[i]}, \dots, f_{n_F}^{[i]}\}$, $i = 1, 2, 3, 4$, with $f_j^{[i]} < f_{j+1}^{[i]}$ (the use, for instance, of f_j^2 to describe elements of F_2 is confusing because it looks like a square. This is the reason why we use the notation $f_j^{[i]}$).

5. HAHN POLYNOMIALS

We start with some basic definitions and facts about Hahn and dual Hahn polynomials, which we will need later. For $a, a + b + 1, a + b + N + 1 \neq -1, -2, \dots$ we write $(h_n^{a,b,N})_n$ for the sequence of Hahn polynomials defined by

$$(5.1) \quad h_n^{a,b,N}(x) = \sum_{j=0}^n \frac{(-x)_j (N - n + 1)_{n-j} (a + b + 1)_{j+n}}{(2 + a + b + N)_n (a + 1)_j (n - j)! j!}, \quad n \geq 0,$$

(we use a different normalization from the one used in [9], pp. 35. The equivalence is given by $\frac{(-1)^n n! (a+1)_n (2+a+b+N)_n}{(a+b+1)_{2n}} h_n^{a,b,N}(x) = \tilde{h}_n^{b+N+1, a+1, N+1}(x)$, where $(\tilde{h}_n^{\alpha, c, N})_n$ is the family used in [9]).

When N is a positive integer then the polynomial $h_n^{a,b,N}(x)$ for $n \geq N + 1$ is always divisible by $(-x)_{N+1}$. Hence

$$(5.2) \quad h_n^{a,b,N}(i) = 0, \quad n \geq N + 1, \quad i = 0, \dots, N.$$

Hahn polynomials are eigenfunctions of the following second-order difference operator

$$(5.3) \quad D_{a,b,N} = x(x-b-N-1)\mathfrak{S}_{-1} - [(x+a+1)(x-N) + x(x-b-N-1)]\mathfrak{S}_0 + (x+a+1)(x-N)\mathfrak{S}_1.$$

That is

$$D_{a,b,N}(h_n^{a,b,N}) = \theta_n h_n^{a,b,N}, \quad \theta_n = n(n + a + b + 1), \quad n \geq 0.$$

They satisfy the following three-term recurrence formula ($h_{-1}^{a,b,N} = 0$)

$$(5.4) \quad xh_n = a_{n+1}h_{n+1} + b_nh_n + c_nh_{n-1}, \quad n \geq 0,$$

where

$$\begin{aligned} a_n &= -\frac{n(n+a)(n+a+b+N+1)}{(2n+a+b-1)(2n+a+b)}, \\ b_n &= \frac{N(a+1)(a+b) + n(2N+b-a)(n+a+b+1)}{(2n+a+b)(2n+a+b+2)}, \\ c_n &= -\frac{(n+a+b)(n+b)(N-n+1)}{(2n+a+b)(2n+a+b+1)} \end{aligned}$$

(to simplify the notation we remove the parameters in some formulas).

Assume that $a, b, a+b, a+b+N+1 \neq -1, -2, \dots$, and $N+1$ is not a positive integer, then the Hahn polynomials are always orthogonal with respect to a moment functional $\rho_{a,b,N}$ which we normalize by taking

$$\langle \rho_{a,b,N}, 1 \rangle = \frac{\Gamma(a+b+N+2)\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}.$$

When $N+1$ is a positive integer, $a, b \neq -1, -2, \dots, -N$, and $a+b \neq -1, -2, \dots, -2N-1$, the first $N+1$ Hahn polynomials are orthogonal with respect to the Hahn measure

$$(5.5) \quad \rho_{a,b,N} = N! \sum_{x=0}^N \frac{\Gamma(a+x+1)\Gamma(N-x+b+1)}{x!(N-x)!} \delta_x,$$

and have non-null norms. The discrete measure $\rho_{a,b,N}$ is positive only when $a, b > -1$ or $a, b < -N$.

We also need the so-called dual Hahn polynomials, $a \neq -1, -2, \dots, n \geq 0$,

$$(5.6) \quad R_n^{a,b,N}(x) = \sum_{j=0}^n \frac{(-1)^j (-n)_j (-N+j)_{n-j}}{(a+1)_j j!} \prod_{i=0}^{j-1} [x - i(i+a+b+1)].$$

Observe that $(-1)^j \prod_{i=0}^{j-1} (x(x+a+b+1) - i(a+b+1+i)) = (-x)_j (x+a+b+1)_j$, therefore we have the duality

$$R_x^{a,b,N}(n(n+a+b+1)) = \frac{(-1)^n n! (N+a+b+2)_n (-N)_x}{(a+b+1)_n (-N)_n} h_n^{a,b,N}(x), \quad x, n \geq 0.$$

Consider now the algebra of differential operators defined by (3.1). There are 4 different \mathcal{D} -operators for the Hahn polynomials (see Lemma 7.2 of [9]). They are defined by the sequences $(\varepsilon_{n,h})_n$ and $(\sigma_n)_n, h = 1, 2, 3, 4$, given by

$$(5.7) \quad \varepsilon_{n,1} = -\frac{n-N+1}{n+a+b+N+1}, \quad \sigma_n = -(2n+a+b-1),$$

$$(5.8) \quad \varepsilon_{n,2} = \frac{(n+b)(n-N+1)}{(n+a)(n+a+b+N+1)}, \quad \sigma_n = -(2n+a+b-1),$$

$$(5.9) \quad \varepsilon_{n,3} = 1, \quad \sigma_n = -(2n+a+b-1),$$

$$(5.10) \quad \varepsilon_{n,4} = -\frac{n+b}{n+a}, \quad \sigma_n = -(2n+a+b-1).$$

These sequences define four \mathcal{D} -operators (see (3.2)):

(5.11)

$$\mathcal{D}_1 = \frac{a+b+1}{2}I + x\nabla, \quad \mathcal{D}_2 = \frac{a+b+1}{2}I + (x-N)\Delta,$$

(5.12)

$$\mathcal{D}_3 = \frac{a+b+1}{2}I + (x+a+1)\Delta, \quad \mathcal{D}_4 = \frac{a+b+1}{2}I + (x-b-N-1)\nabla.$$

Δ and ∇ denote, as usual, the first-order difference operators:

$$\Delta(f) = f(x+1) - f(x), \quad \nabla(f) = f(x) - f(x-1).$$

Let us call $N_x^{h;j}$ and $D_x^{h;j}$, $h = 1, 2$, the following functions:

$$\begin{aligned} N_x^{1;j} &= (x-j+b+1)_j & N_x^{2;j} &= (x-j-N)_j, \\ D_x^{1;j} &= (-1)^j(x-j+a+1)_j & D_x^{2;j} &= (-1)^j(x-j+a+b+N+2)_j. \end{aligned}$$

We will use the following properties, which easily hold by definition

$$(5.13) \quad N_{x-i}^{h;m-i} = N_{x-i}^{h;j} N_{x-i-j}^{h;m-i-j}, \quad D_{x-i}^{h;m-i} = D_{x-i}^{h;j} D_{x-i-j}^{h;m-i-j}, \quad h = 1, 2.$$

Given a quartet $\mathcal{U} = (U_1, U_2, U_3, U_4)$ of finite sets of nonnegative integers, we write m_j for the cardinal of U_j , $j = 1, 2, 3, 4$, $m = m_1 + m_2 + m_3 + m_4$ and

(5.14)

$$\begin{aligned} \mathbb{U}_1 &= \{1, \dots, m_1\}, & \mathbb{U}_2 &= \{m_1+1, \dots, m_1+m_2\}, \\ \mathbb{U}_3 &= \{m_1+m_2+1, \dots, m_1+m_2+m_3\}, & \mathbb{U}_4 &= \{m_1+m_2+m_3+1, \dots, m\}. \end{aligned}$$

We write $U_j = \{u_i^j, i \in \mathbb{U}_j\}$ (see the end of Section 4.2 for a justification of this notation).

The functions $\xi_{x,j}^h$ defined in (3.3) can be written as

(5.15)

$$\xi_{x,j}^h = \begin{cases} \frac{N_x^{2;j}}{D_x^{2;j}} = (-1)^j \frac{(x-j-N)_j}{(x-j+a+b+N+2)_j}, & \text{for } h \in \mathbb{U}_1, \\ \frac{N_x^{1;j} N_x^{2;j}}{D_x^{1;j} D_x^{2;j}} = \frac{(x-j+b+1)_j (x-j-N)_j}{(x-j+a+1)_j (x-j+a+b+N+2)_j}, & \text{for } h \in \mathbb{U}_2, \\ 1, & \text{for } h \in \mathbb{U}_3, \\ \frac{N_x^{1;j}}{D_x^{1;j}} = (-1)^j \frac{(x-j+b+1)_j}{(x-j+a+1)_j}, & \text{for } h \in \mathbb{U}_4. \end{cases}$$

In the rest of this Section, we will prove that the three hypotheses (3.8), (3.9) and (3.10) in Theorem 3.1 hold for the four \mathcal{D} -operators above associated to the Hahn polynomials. Hence, the polynomials $(q_n)_n$ defined by (3.5) for the Hahn family will be consequently eigenfunctions of the higher-order difference operator (3.11).

To check the first hypothesis (3.8) in Theorem 3.1, we will need the following lemma, which it will be also useful to compute the order of the difference operator with respect to which the bispectral polynomials constructed in Section 6 will be eigenfunctions.

Lemma 5.1. Let Y_1, Y_2, \dots, Y_m , be nonzero polynomials satisfying that $\deg Y_i = u_i^j$, if $i \in \mathbb{U}_j$ and $1 \leq j \leq 4$. Write r_i for the leading coefficient of Y_i , $1 \leq i \leq m$. For real numbers a, b, N , consider the rational function P defined by

$$(5.16) \quad P(x) = \frac{\begin{vmatrix} & & & & j=1, \dots, m \\ \left[\begin{array}{c} N_{x-j}^{2;m-j} D_{x-1}^{2;j-1} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_1 \end{array} \right] & & & & \\ \left[\begin{array}{c} N_{x-j}^{1;m-j} N_{x-j}^{2;m-j} D_{x-1}^{1;j-1} D_{x-1}^{2;j-1} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_2 \end{array} \right] & & & & \\ \left[\begin{array}{c} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_3 \end{array} \right] & & & & \\ \left[\begin{array}{c} N_{x-j}^{1;m-j} D_{x-1}^{1;j-1} Y_i(\theta_{x-j}) \\ i \in \mathbb{U}_4 \end{array} \right] & & & & \end{vmatrix}}{p(x)q(x)},$$

where p and q are the polynomials

$$(5.17) \quad p(x) = \prod_{i=1}^{m_2+m_4-1} N_{x-m_1-m_3-i}^{1;m_2+m_4-i} D_{x-1}^{1;m_2+m_4-i} \prod_{i=1}^{m_1+m_2-1} N_{x-m_3-m_4-i}^{2;m_1+m_2-i} D_{x-1}^{2;m_1+m_2-i},$$

$$(5.18) \quad q(x) = (-1)^{\frac{m(m-1)}{2}} \prod_{p=1}^{m-1} \left(\prod_{s=1}^p \sigma_{x-m+\frac{s+p+1}{2}} \right).$$

The determinant (5.16) should be understood in the way explained in the Preliminaries (see (2.1)). If

$$(5.19) \quad z - u + b + N + 1, w - v + a + N + 1 \neq 0,$$

for $u \in U_1, v \in U_2, w \in U_3, z \in U_4$, then P is a polynomial of degree

$$d = 2 \sum_{u \in U_1, U_2, U_3, U_4} u - 2 \sum_{i=1}^4 \binom{m_i}{2},$$

with leading coefficient given by

$$r = (-1)^{\sum_{i=1}^4 \binom{m_i}{2} + m_1 m_2 + m_2 m_3 + m_3 m_4} V_{U_1} V_{U_2} V_{U_3} V_{U_4} \prod_{i=1}^m r_i \\ \times \prod_{v \in U_2, w \in U_3} (N + a + 1 - v + w) \prod_{u \in U_1, z \in U_4} (N + b + 1 - u + z),$$

where V_X denotes the Vandermonde determinant associated to the set $X = \{x_1, x_2, \dots, x_K\}$ defined by $V_X = \prod_{i < j} (x_j - x_i)$.

Proof. The Lemma can be proved using the same approach as in the proof of Lemma 3.3 in [10]. \square

Let us now introduce the key concept in order to check the hypotheses (3.9) and (3.10) in Theorem 3.1 for the Hahn polynomials. We define an *involution* that characterizes the subring $\mathbb{R}[\theta_x]$ in $\mathbb{R}[x]$. This involution is given by

$$(5.20) \quad \mathcal{I}^{a+b}(p(x)) = p(-(x+a+b+1)), \quad p \in \mathbb{R}[x].$$

Clearly we have $\mathcal{I}^{a+b}(\theta_x) = \theta_x$. Hence every polynomial in θ_x is invariant under the action of \mathcal{I}^{a+b} . Conversely, if $p \in \mathbb{R}[x]$ is invariant under \mathcal{I}^{a+b} , then $p \in \mathbb{R}[\theta_x]$. We also have that if $p \in \mathbb{R}[x]$ is skew invariant, i.e. $\mathcal{I}^{a+b}(p) = -p$, then p is divisible by $\theta_{x-1/2} - \theta_{x+1/2}$, and the quotient belongs to $\mathbb{R}[\theta_x]$. We remark here that, in the case of Hahn polynomials and from the definition of θ_x and σ_x , we have that $\sigma_{x+1} = \theta_{x-1/2} - \theta_{x+1/2}$. Observe that σ_{x+1} is skew invariant itself.

We have the following properties according to the definition (5.20), $h = 1, 2$:

$$(5.21) \quad \mathcal{I}^{a+b+i}(\theta_{x-j}) = \theta_{x+i+j}, \quad \mathcal{I}^{a+b+i}(\sigma_{x-j}) = -\sigma_{x+i+j+2},$$

$$(5.22) \quad \mathcal{I}^{a+b+i}(N_{x-j-s}^{h;m-s}) = D_{x+m+i+j}^{h;m-s}, \quad \mathcal{I}^{a+b+i}(D_{x-j-s}^{h;m-s}) = N_{x+m+i+j}^{h;m-s}.$$

We are now ready to check that the three hypotheses (3.8), (3.9) and (3.10) in Theorem 3.1 hold for the four \mathcal{D} -operators above associated to the Hahn polynomials.

Lemma 5.2. *Let \mathcal{A} and $(p_n)_n$ be respectively, the algebra of difference operators (3.1) and the sequence of Hahn polynomials $p_n = h_n^{a,b,N}$. We denote by D_p the second-order difference operator (5.3), so that $\theta_n = n(n+a+b+1)$ and $D_p(p_n) = \theta_n p_n$. For $j = 1, 2, 3, 4$, we also have m_j \mathcal{D} -operators defined by the sequences $(\varepsilon_{n,j})_n$, $(\sigma_n)_n$ (see (5.7)–(5.10)). Write then $m = \sum_{i=1}^4 m_i$ and let Ξ be a polynomial in x invariant under the action of $\mathcal{I}^{a+b-m-1}$. Define the rational function S by*

$$(5.23) \quad S(x) = \sigma_{x-\frac{m-1}{2}} \Xi(x) \frac{(D_{x-1}^{1;m-1})^{m_2+m_4} (D_{x-1}^{2;m-1})^{m_1+m_2}}{p(x)q(x)},$$

where p and q are the polynomials defined by (5.17) and (5.18), respectively. Then the three hypotheses (3.8), (3.9) and (3.10) in Theorem 3.1 hold.

Proof. Consider the sets \mathbb{U}_j , $j = 1, 2, 3, 4$, given by (5.14). By interchanging rows, we can assume that each \mathbb{U}_j is formed by the indexes h where the \mathcal{D} -operator \mathcal{D}_h is defined by the sequence $(\varepsilon_{n,j})_n$ (see (5.7)–(5.10)).

Since the polynomial Ξ is invariant under the action of $\mathcal{I}^{a+b-m-1}$, we have

$$(5.24) \quad \mathcal{I}^{a+b+i}(\Xi(x-j)) = \Xi(x+m+i+j+1).$$

As a consequence of (5.21) and (5.22) we have

$$(5.25) \quad \mathcal{I}^{a+b+i}(q(x-j)) = (-1)^{\frac{m(m-1)}{2}} q(x+i+j+m+1),$$

and

$$(5.26) \quad \mathcal{I}^{a+b+i}(p(x-j)) = p(x+i+j+m+1).$$

We now check the first assumption (3.8) in Theorem 3.1, that is: $S(x)\Omega(x)$ is a polynomial in x .

From the definition of $S(x)$ in (5.23) and $\Omega(x)$ in (3.4) it is straightforward to see, using (5.15) and (5.13), that

(5.27)

$$S(x)\Omega(x) = \frac{\sigma_{x-\frac{m-1}{2}}\Xi(x)}{p(x)q(x)} \left| \begin{array}{c} \begin{array}{c} j=1, \dots, m \\ \left[\begin{array}{c} N_{x-j}^{2;m-j} D_{x-1}^{2;j-1} Y_l(\theta_{x-j}) \\ \mathcal{U} \in \mathbb{U}_1 \end{array} \right] \\ \left[\begin{array}{c} N_{x-j}^{1;m-j} N_{x-j}^{2;m-j} D_{x-1}^{1;j-1} D_{x-1}^{2;j-1} Y_l(\theta_{x-j}) \\ \mathcal{U} \in \mathbb{U}_2 \end{array} \right] \\ \left[\begin{array}{c} Y_l(\theta_{x-j}) \\ \mathcal{U} \in \mathbb{U}_3 \end{array} \right] \\ \left[\begin{array}{c} N_{x-j}^{1;m-j} D_{x-1}^{1;j-1} Y_l(\theta_{x-j}) \\ \mathcal{U} \in \mathbb{U}_4 \end{array} \right] \end{array} \right|.$$

Therefore $S(x)\Omega(x) = \sigma_{x-\frac{m-1}{2}}\Xi(x)P(x)$, where P is the rational function (5.16) defined in the Lemma 5.1. According to this Lemma, P is actually a polynomial and hence $S(x)\Omega(x)$ is a polynomial as well.

We now check the second assumption (3.9) in Theorem 3.1, that is: there exist $\tilde{M}_1, \dots, \tilde{M}_m$, polynomials in x such that

$$M_h(x) = \sigma_{x+1}\tilde{M}_h(\theta_x), \quad h = 1, \dots, m.$$

Write now

$$\Psi_j^h(x) = \xi_{x-j, m-j}^h S(x) \det (\xi_{x-r, m-r}^l Y_l(\theta_{x-r}))_{l \in \mathbb{I}_h; r \in \mathbb{I}_j}, \quad h, j = 1, \dots, m.$$

A simple computation using (5.15) shows that Ψ_j^h , $h, j = 1, \dots, m$, are polynomials in x . Hence Lemma 3.4 gives that M_h is also a polynomial in x .

It is now enough to see that

$$\mathcal{I}^{a+b}(M_h(x)) = -M_h(x), \quad h = 1, \dots, m,$$

where $M_h(x)$, $h = 1, \dots, m$, are defined in (3.7). Hence, $M_h(x)$, $h = 1, \dots, m$, according to the discussion after (5.20), is divisible by σ_{x+1} and the quotient belongs to $\mathbb{R}[\theta_x]$.

Assume that the h -th \mathcal{D} -operator is \mathcal{D}_1 (similar for \mathcal{D}_2 , \mathcal{D}_3 and \mathcal{D}_4). In that case, as before, we can remove all denominators in $M_h(x)$ and rearrange the determinant to write

$$M_h(x) = \sum_{j=1}^m (-1)^{h+j} \frac{\sigma_{x+j-\frac{m-1}{2}}\Xi(x+j)}{p(x+j)q(x+j)} N_x^{2;m-j} D_{x+j-1}^{2;j-1} \times$$

$$\times \left[\begin{array}{c} \left[N_{x+j-r}^{2;m-r} D_{x+j-1}^{2;r-1} Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_1 \setminus \{h\} \\ \left[N_{x+j-r}^{1;m-r} N_{x+j-r}^{2;m-r} D_{x+j-1}^{1;r-1} D_{x+j-1}^{2;r-1} Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_2 \\ \left[Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_3 \\ \left[N_{x+j-r}^{1;m-r} D_{x+j-1}^{1;r-1} Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_4 \end{array} \right]^{l \neq h, r \neq j}.$$

Hence, using (5.22), (5.24), (5.25) and (5.26), we have

$$\begin{aligned} \mathcal{I}^{a+b}(M_h(x)) &= - \sum_{j=1}^m (-1)^{h+j} (-1)^{\frac{m(m-1)}{2}} \frac{\sigma_{x+\frac{m-1}{2}-j+2} \Xi(x+m-j+1)}{p(x+m-j+1)q(x+m-j+1)} \times \\ &\quad \times D_{x+m-j}^{2;m-j} N_x^{2;j-1} \left[\begin{array}{c} \left[D_{x+m-j}^{2;m-r} N_{x+r-j}^{2;r-1} Y_l(\theta_{x-j+r}) \right] \\ l \in \mathbb{U}_1 \setminus \{h\} \\ \left[D_{x+m-j}^{1;m-r} D_{x+m-j}^{2;m-r} N_{x+r-j}^{1;r-1} N_{x+r-j}^{2;r-1} Y_l(\theta_{x-j+r}) \right] \\ l \in \mathbb{U}_2 \\ \left[Y_l(\theta_{x-j+r}) \right] \\ l \in \mathbb{U}_3 \\ \left[D_{x+m-j}^{1;m-r} N_{x+r-j}^{1;r-1} Y_l(\theta_{x-j+r}) \right] \\ l \in \mathbb{U}_4 \end{array} \right]^{l \neq h, r \neq j} \\ &= -(-1)^{\frac{m(m-1)}{2}} (-1)^{m-1} (-1)^{\frac{(m-1)(m-2)}{2}} \sum_{j=1}^m (-1)^{h+j} \frac{\sigma_{x+j-\frac{m-1}{2}} \Xi(x+j)}{p(x+j)q(x+j)} \times \\ &\quad \times D_{x+j-1}^{2;j-1} N_x^{2;m-j} \left[\begin{array}{c} \left[D_{x+j-1}^{2;r-1} N_{x+j-r}^{2;m-r} Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_1 \setminus \{h\} \\ \left[D_{x+j-1}^{1;r-1} D_{x+j-1}^{2;r-1} N_{x+j-r}^{1;m-r} N_{x+j-r}^{2;m-r} Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_2 \\ \left[Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_3 \\ \left[D_{x+j-1}^{1;r-1} N_{x+j-r}^{1;m-r} Y_l(\theta_{x+j-r}) \right] \\ l \in \mathbb{U}_4 \end{array} \right]^{l \neq h, r \neq j} \end{aligned}$$

$$= -(-1)^{m(m-1)} M_h(x) = -M_h(x).$$

Observe that in the second step we have renamed the index j while in the determinant we have interchanged all columns, so we have the corresponding change of signs.

We finally check the third assumption (3.10) in Theorem 3.1, that is: there exists a polynomial P_S such that

$$P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^m Y_h(\theta_x) M_h(x).$$

Call

$$H(x) = 2\lambda_x + \sum_{h=1}^m Y_h(\theta_x) M_h(x).$$

Taking into account that M_h , $h = 1, \dots, m$, are polynomials in x , we conclude that H is also a polynomial in x . From (3.12) we have that

$$H(x) - H(x-1) = S(x)\Omega(x) + S(x+m)\Omega(x+m).$$

For this step it is enough to see that

$$(5.28) \quad \mathcal{I}^{a+b-1}(H(x) - H(x-1)) = -(H(x) - H(x-1)).$$

If such is the case, then we can always find a polynomial P_S such that $P_S(\theta_x) = H(x)$, since we already know, from (5.21), that

$$\mathcal{I}^{a+b-1}(P_S(\theta_x) - P_S(\theta_{x-1})) = P_S(\theta_{x-1}) - P_S(\theta_x) = -(P_S(\theta_x) - P_S(\theta_{x-1})).$$

To proof (5.28) it is enough to see that

$$\mathcal{I}^{a+b-1}(S(x)\Omega(x)) = -(S(x+m)\Omega(x+m)).$$

From (5.27) we have, using again (5.22), (5.24), (5.25) and (5.26), that

$$\begin{aligned} \mathcal{I}^{a+b-1}(S(x)\Omega(x)) &= (-1)^{\frac{m(m-1)}{2}} \frac{\sigma_{x+\frac{m+1}{2}} \Xi(x+m)}{p(x+m)q(x+m)} \times \\ &\quad \times \left| \begin{array}{c} \begin{array}{c} \left[D_{x+m-1}^{2;m-j} N_{x+j-1}^{2;j-1} Y_l(\theta_{x+j-1}) \right] \\ \mathcal{I} \in \mathbb{U}_1 \end{array} \\ \begin{array}{c} \left[D_{x+m-1}^{1;m-j} D_{x+m-1}^{2;m-j} N_{x+j-1}^{1;j-1} N_{x+j-1}^{2;j-1} Y_l(\theta_{x+j-1}) \right] \\ \mathcal{I} \in \mathbb{U}_2 \end{array} \\ \begin{array}{c} \left[Y_l(\theta_{x+j-1}) \right] \\ \mathcal{I} \in \mathbb{U}_3 \end{array} \\ \begin{array}{c} \left[D_{x+m-1}^{1;m-j} N_{x+j-1}^{1;j-1} Y_l(\theta_{x+j-1}) \right] \\ \mathcal{I} \in \mathbb{U}_4 \end{array} \end{array} \right|^{j=1, \dots, m} \\ &= -(-1)^{\frac{m(m-1)}{2}} (-1)^{\frac{m(m-1)}{2}} \frac{\sigma_{x+m-\frac{m-1}{2}} \Xi(x+m)}{p(x+m)q(x+m)} \times \end{aligned}$$

$$\begin{aligned}
& \times \left[\begin{array}{c} \left[D_{x+m-1}^{2;j-1} N_{x+m-j}^{2;m-j} Y_l(\theta_{x+m-j}) \right] \\ l \in \mathbb{U}_1 \end{array} \right] \\
& \left[\begin{array}{c} \left[D_{x+m-1}^{1;j-1} D_{x+m-1}^{2;j-1} N_{x+m-j}^{1;m-j} N_{x+m-j}^{2;m-j} Y_l(\theta_{x+m-j}) \right] \\ l \in \mathbb{U}_2 \end{array} \right] \\
& \left[\begin{array}{c} \left[Y_l(\theta_{x+m-j}) \right] \\ l \in \mathbb{U}_3 \end{array} \right] \\
& \left[\begin{array}{c} \left[D_{x+m-1}^{1;j-1} N_{x+m-j}^{1;m-j} Y_l(\theta_{x+m-j}) \right] \\ l \in \mathbb{U}_4 \end{array} \right] \\
& = -S(x+m)\Omega(x+m).
\end{aligned}$$

□

6. BISPECTRAL KRALL-HAHN POLYNOMIALS

In this Section we put together all the ingredients showed in the previous Sections to construct bispectral Krall-Hahn polynomials.

We can apply Theorem 3.1 to produce from arbitrary polynomials Y_j , $j \geq 0$, a large class of sequences of polynomials $(q_n)_n$ satisfying higher-order difference equations. But only for a convenient choice of the polynomials Y_j , $j \geq 0$, these polynomials $(q_n)_n$ are also orthogonal with respect to a measure. As we wrote in the Introduction, when the sequence $(p_n)_n$ is the Hahn polynomials, a very nice symmetry between the family $(p_n)_n$ and the polynomials Y_j 's appears. Indeed, the polynomials Y_j can be chosen as dual Hahn polynomials with parameters depending on the \mathcal{D} -operator \mathcal{D}_h . This symmetry is given by the recurrence relation (4.2), where $Y_i = Z_i^h$.

Lemma 6.1. *Consider the dual Hahn polynomials*

$$\begin{aligned}
Z_j^1(x) &= R_j^{-b, -a, a+b+N}(x+a+b), \quad j \geq 0, \\
Z_j^2(x) &= R_j^{-a, -b, a+b+N}(x+a+b), \quad j \geq 0, \\
Z_j^3(x) &= R_j^{-b, -a, -2-N}(x+a+b), \quad j \geq 0, \\
Z_j^4(x) &= R_j^{-a, -b, -2-N}(x+a+b), \quad j \geq 0.
\end{aligned}$$

Then they satisfy the recurrence (4.2), where $(a_n)_{n \in \mathbb{Z}}$, $(b_n)_{n \in \mathbb{Z}}$, $(c_n)_{n \in \mathbb{Z}}$ are the sequences of coefficients in the three-term recurrence relation for the Hahn polynomials $(h_n^{a,b,N})_n$ (5.4) and

$$\begin{aligned}
\varepsilon_n^1 &= -\frac{n-N-1}{n+a+b+N+1}, & \eta_1 &= 1, & \kappa_1 &= -b-N, \\
\varepsilon_n^2 &= \frac{(n-N-1)(n+b)}{(n+a)(n+a+b+N+1)}, & \eta_2 &= -1, & \kappa_2 &= a, \\
\varepsilon_n^3 &= 1, & \eta_3 &= -1, & \kappa_3 &= -N-1, \\
\varepsilon_n^4 &= -\frac{n+b}{n+a}, & \eta_4 &= 1, & \kappa_4 &= 1.
\end{aligned}$$

Proof. It is similar to the proof of Lemma 5.1 of [11]. \square

From now on we will assume that N is a positive integer. This condition is necessary for the existence of a positive weight for the Hahn polynomials, and only in this case we have an explicit expression of that weight. However, this condition is not needed in our construction and hence Theorem 6.2 and Corollary 6.3 are also valid when N is not a positive integer (once one has adapted the constraints on the parameters a and b).

Since we have four \mathcal{D} -operators for Hahn polynomials, we make a partition of the indices in Theorem 3.1 (see (5.14)) and take

$$(6.1) \quad \varepsilon_n^h = \begin{cases} -\frac{n-N-1}{n+a+b+N+1}, & \text{for } h \in \mathbb{U}_1, \\ \frac{(n-N-1)(n+b)}{(n+a)(n+a+b+N+1)}, & \text{for } h \in \mathbb{U}_2, \\ 1, & \text{for } h \in \mathbb{U}_3, \\ -\frac{n+b}{n+a}, & \text{for } h \in \mathbb{U}_4. \end{cases}$$

For the sequences $(\sigma_n^h)_n$, we always take $\sigma_n^h = \sigma_n = -(2n+a+b-1)$. In particular, the auxiliary functions $\xi_{x,i}^h$, $h = 1, \dots, m$, $i \in \mathbb{Z}$, (see (3.3)) are then defined by (5.15). Finally write

$$\mathbb{Z}_i = \{j \in \mathbb{Z} : j \leq i\}, \quad i \in \mathbb{Z}.$$

We are now ready to establish the main Theorem of this paper.

Theorem 6.2. *Let $\mathcal{F} = (F_1, F_2, F_3, F_4)$ be a quartet of finite sets of positive integers (the empty set is allowed, in which case we take $\max F = -1$). For $\mathfrak{h} = (h_1, h_2, h_3)$, $h_i \geq 1$, consider the quartet $\mathcal{U} = (U_1, U_2, U_3, U_4)$ whose elements are the transformed sets $J_{h_j}(F_j) = U_j = \{u_i^{j\uparrow} : i \in \mathbb{U}_j\}$, $j = 1, 2, 3$, and $I(F_4) = U_4 = \{u_i^{4\uparrow} : i \in \mathbb{U}_4\}$, where the involution I and the transform J_h are defined by (4.8) and (4.9), respectively. Define $m = m_1 + m_2 + m_3 + m_4$, where m_i denotes the cardinal of U_i , $f_{i,M} = \max(F_i)$, and n_{F_i} the cardinal of F_i for $i = 1, 2, 3, 4$. Let a and b be real numbers satisfying*

$$(6.2) \quad a \notin \mathbb{Z}_{f_2, M+f_4, M+h_2}, \quad b \notin \mathbb{Z}_{f_1, M+f_3, M+h_1+h_3-1}, \quad a+b \notin \mathbb{Z}_{\sum_{i=1}^4 f_{i, M} + \sum_{i=1}^3 h_i}.$$

In addition, we assume that

$$(6.3) \quad a \neq 1, 2, \dots, \quad \text{if } F_2 \text{ or } F_4 \neq \emptyset, \quad \text{and} \quad b \neq 1, 2, \dots, \quad \text{if } F_1 \text{ or } F_3 \neq \emptyset.$$

Consider the Hahn and dual Hahn polynomials $(h_n^{a,b,N})_n$ (5.1) and $(R_n^{a,b,N})_n$ (5.6), respectively. Assume that $\Omega_{a,b,N}^{\mathcal{U}, \mathfrak{h}}(n) \neq 0$ for $0 \leq n \leq N + m_3 + m_4 + 1$ where the

$m \times m$ Casorati determinant $\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}$ is defined by

$$\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(x) = \begin{vmatrix} & & & & j=1,\dots,m \\ \begin{bmatrix} \xi_{x-j,m-j}^1 R_u^{-b,-a,a+b+N}(\theta_{x-j} + a + b) \\ u \in U_1 \end{bmatrix} \\ \begin{bmatrix} \xi_{x-j,m-j}^2 R_u^{-a,-b,a+b+N}(\theta_{x-j} + a + b) \\ u \in U_2 \end{bmatrix} \\ \begin{bmatrix} \xi_{x-j,m-j}^3 R_u^{-b,-a,-2-N}(\theta_{x-j} + a + b) \\ u \in U_3 \end{bmatrix} \\ \begin{bmatrix} \xi_{x-j,m-j}^4 R_u^{-a,-b,-2-N}(\theta_{x-j} + a + b) \\ u \in U_4 \end{bmatrix} \end{vmatrix}.$$

We then define the sequence of polynomials q_n , $n \geq 0$, by

$$(6.4) \quad q_n(x) = \begin{vmatrix} & & & & j=1,\dots,m+1 \\ (-1)^{j-1} h_{n+1-j}^{a,b,N}(x) \\ \begin{bmatrix} \xi_{n-j,m-j}^1 R_u^{-b,-a,a+b+N}(\theta_{n-j} + a + b) \\ u \in U_1 \end{bmatrix} \\ \begin{bmatrix} \xi_{n-j,m-j}^2 R_u^{-a,-b,a+b+N}(\theta_{n-j} + a + b) \\ u \in U_2 \end{bmatrix} \\ \begin{bmatrix} \xi_{n-j,m-j}^3 R_u^{-b,-a,-2-N}(\theta_{n-j} + a + b) \\ u \in U_3 \end{bmatrix} \\ \begin{bmatrix} \xi_{n-j,m-j}^4 R_u^{-a,-b,-2-N}(\theta_{n-j} + a + b) \\ u \in U_4 \end{bmatrix} \end{vmatrix}.$$

Then

(1) The polynomials q_n , $0 \leq n \leq N + m_3 + m_4$, are orthogonal and have non-null norms with respect to the measure

$$\begin{aligned} \tilde{\rho}_{a,b,N}^{\mathcal{F},\mathfrak{h}} &= \prod_{f \in F_1} (b + N + 1 - f - x) \prod_{f \in F_2} (x + a + 1 - f) \\ &\quad \times \prod_{f \in F_3} (N + f - x) \prod_{f \in F_4} (x + f_{4,M} + 1 - f) \rho_{\tilde{a},\tilde{b},\tilde{N}}(x + f_{4,M} + 1), \end{aligned}$$

where

$$\tilde{a} = a - f_{2,M} - f_{4,M} - h_2 - 1, \quad \tilde{b} = b - f_{1,M} - f_{3,M} - h_1 - h_3, \quad \tilde{N} = N + f_{3,M} + f_{4,M} + h_3 + 1,$$

and $\rho_{a,b,N}$ is the Hahn weight (5.5).

(2) The polynomials q_n , $0 \leq n \leq N + m_3 + m_4$, are eigenfunctions of a higher-order difference operator of the form (1.1) with

$$-s = r = \sum_{f \in F_4} f - \sum_{f \in F_1, F_2, F_3} f - \sum_{i=1}^4 \binom{n_{F_i}}{2} + \sum_{i=1}^3 n_{F_i} (f_{i,M} + h_i) + 1$$

(which can be constructed using Theorem 3.1).

Proof. Notice that the assumption (6.2) on the parameters a and b implies that

$$\tilde{a}, \tilde{b} \neq -1, \dots, -\tilde{N}, \tilde{a} + \tilde{b} \neq -1, \dots, -2\tilde{N} - 1,$$

and hence the Hahn weight $\rho_{\tilde{a}, \tilde{b}, \tilde{N}}(x + f_{4,M} + 1)$ is well defined and its support is $\{-f_{4,M} - 1, \dots, N + f_{3,M} + h_3\}$. Using the assumptions (6.2) on the parameters a and b , we deduce that the support of the measure $\tilde{\rho}_{a,b,N}^{\mathcal{F},h}$ is

$$\{-f_{4,M} - 1, \dots, N + f_{3,M} + h_3\} \setminus \left((N + F_3) \cup (-f_{4,M} - 1 + F_4) \right).$$

Notice that the support is formed by $N + f_{3,M} + f_{4,M} + h_3 + 2 - n_{F_3} - n_{F_4}$ integers. Taking into account that $U_3 = J_{h_3}(F_3)$, $U_4 = I(F_4)$ and (4.10) we get

$$N + f_{3,M} + f_{4,M} + h_3 + 2 - n_{F_3} - n_{F_4} = N + m_3 + m_4 + 1.$$

Before going on with the proof we comment on the assumption that $\Omega_{a,b,N}^{\mathcal{U},h}(n) \neq 0$ for $0 \leq n \leq N + m_3 + m_4 + 1$. If $F_1 \neq \emptyset$ or $F_2 \neq \emptyset$, since the sequence ε_n^h , $h \in \mathbb{U}_1 \cup \mathbb{U}_2$, vanish for $n = N + 1$, it is not difficult to see that $\Omega_{a,b,N}^{\mathcal{U},h}(n) = 0$ for $N + m_3 + m_4 + 2 \leq n \leq N + m$. If $F_1 = \emptyset$ and $F_2 = \emptyset$, the situation is different, and, except for exceptional values of the parameters a, b and N , we have $\Omega_{a,b,N}^{\mathcal{U},h}(n) \neq 0$ for all $n \geq 0$. In this case, the polynomials $(q_n)_n$ are defined for all $n \geq 0$ and always have degree n . However, it is not difficult to see that for $n \geq N + m_3 + m_4 + 1$, the polynomial $q_n(x)$ vanishes in the support of $\tilde{\rho}_{a,b,N}^{\mathcal{F},h}$. Hence they are still orthogonal with respect to this measure but have null norms. This is completely analogous to the situation with the Hahn polynomials $h_n^{a,b,N}$, which are defined for all $n \geq 0$ (except when $a, a + b + 1 = -1, -2, \dots$) and always have degree n . But if $n \geq N + 1$, they vanish in the support of its weight (see (5.2)).

To prove (1) of the Theorem, we use the strategy of the Section 4.1.

We need some notation. Write Z_j^h , $h = 1, \dots, m, j \geq 0$, for the polynomials

$$(6.5) \quad Z_j^h(x) = \begin{cases} R_j^{-b, -a, a+b+N}(x + a + b), & h \in \mathbb{U}_1, \\ R_j^{-a, -b, a+b+N}(x + a + b), & h \in \mathbb{U}_2, \\ R_j^{-b, -a, -2-N}(x + a + b), & h \in \mathbb{U}_3, \\ R_j^{-a, -b, -2-N}(x + a + b), & h \in \mathbb{U}_4. \end{cases}$$

The assumptions (6.3) on the parameters a and b implies that these dual Hahn polynomials are well defined and have degree j . Denote by G and \tilde{G} the m -tuples

$$(6.6) \quad G = (u_1^{1\uparrow}, \dots, u_{m_1}^{1\uparrow}, u_1^{2\uparrow}, \dots, u_{m_2}^{2\uparrow}, u_1^{3\uparrow}, \dots, u_{m_3}^{3\uparrow}, u_1^{4\uparrow}, \dots, u_{m_4}^{4\uparrow}) = (g_1, \dots, g_m), \\ \tilde{G} = (\tilde{g}_1, \dots, \tilde{g}_m),$$

where

$$\tilde{g}_i = \begin{cases} u_i^{1\uparrow} - b - N, & i \in \mathbb{U}_1, \\ -u_i^{2\uparrow} + a, & i \in \mathbb{U}_2, \\ -u_i^{3\uparrow} - N - 1, & i \in \mathbb{U}_3, \\ u_i^{4\uparrow} + 1, & i \in \mathbb{U}_4. \end{cases}$$

Finally, write $\mathfrak{p}_{\tilde{G}}$ for the polynomial

$$\mathfrak{p}_{\tilde{G}}(x) = \prod_{u \in U_1} (x - u + b + N) \prod_{u \in U_2} (x + u - a) \prod_{u \in U_3} (x + u + N + 1) \prod_{u \in U_4} (x - u - 1).$$

It is easy to see that $\mathfrak{p}_{\tilde{G}}$ has simple roots if and only if

$$(6.7) \quad \begin{aligned} &u + v - a - b - N, u + w - b + 1, v + z - a + 1, w + z + N + 2 \neq 0, \\ &z - u + b + N + 1, w - v + a + N + 1 \neq 0, \end{aligned}$$

for $u \in U_1, v \in U_2, w \in U_3, z \in U_4$. These constraints follow easily from the assumptions (6.2) on the parameters a and b . Hence, $\mathfrak{p}_{\tilde{G}}$ has simple roots.

Proceeding as in the proof of Theorem 1.1 in [11], one can prove that

$$\begin{aligned} c_{a,b,N}^{\mathcal{F},\mathfrak{h}}(\tilde{\rho}_{a,b,N}^{\mathcal{F},\mathfrak{h}}, h_n^{a,b,N}) &= (-1)^n \sum_{i=1}^m \frac{\xi_{n,n+1}^i Z_{g_i}^i(n)}{\mathfrak{p}'_{\tilde{G}}(\tilde{g}_i) Z_{g_i}^i(-1)}, \quad n \geq 0, \\ \sum_{i=1}^m \frac{Z_{g_i}^i(n)}{\xi_{-1,-n-1}^i \mathfrak{p}'_{\tilde{G}}(\tilde{g}_i) Z_{g_i}^i(-1)} &= 0, \quad 1 - m \leq n < 0, \\ \sum_{i=1}^m \frac{Z_{g_i}^i(n)}{\xi_{-1,m-1}^i \mathfrak{p}'_{\tilde{G}}(\tilde{g}_i) Z_{g_i}^i(-1)} &\neq 0, \end{aligned}$$

where $c_{a,b,N}^{\mathcal{F},\mathfrak{h}}$ is the constant independent of n given by

$$c_{a,b,N}^{\mathcal{F},\mathfrak{h}} = (-1)^{m_2+m_4} \Gamma(a+1) \Gamma(b)(a+b+1)_{N+1} (N+2)_{f_{3,M}+f_{4,M}+h_3},$$

and $\xi_{x,y}^i$ are defined by (5.15).

From the recurrence relation for the Hahn polynomials, we get that $a_n c_n \neq 0$ for $0 \leq n \leq N$, but $c_{N+1} = 0$. It is also easy to check that $\varepsilon_n^h \neq 0$, $h = 1, \dots, m$, when n is a negative integer. Since we assume that $\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(n) \neq 0$ for $0 \leq n \leq N + m_3 + m_4 + 1$, the orthogonality of the polynomials q_n , $0 \leq n \leq N + m_3 + m_4$, with respect to $\tilde{\rho}_{a,b,N}^{\mathcal{F},\mathfrak{h}}$ is now a consequence of the Lemmas 6.1, 4.1 and the identities (6.1) and (5.15). They have also non-null norms.

We now prove (2) of the Theorem. Using (5.15), it is straightforward to see that $\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(x)$ coincides with the (quasi) Casorati determinant

$$\det (\xi_{x-j,m-j}^l Y_l(\theta_{x-j}))_{l,j=1}^m,$$

where $Y_l(x) = Z_{g_l}^l(x)$, and Z_j^l , $l = 1, \dots, m$, $j \geq 0$, and $G = \{g_1, \dots, g_m\}$ are defined by (6.5) and (6.6), respectively. Consider the particular case of the polynomial P (5.16) in Lemma 5.1 for $Y_l(x) = Z_{g_l}^l(x)$ (and denote it again by P), and write S for the rational function

$$S(x) = \sigma_{x-\frac{m-1}{2}} \frac{(D_{x-1}^{1;m-1})_{m_2+m_4} (D_{x-1}^{2;m-1})_{m_1+m_2}}{p(x)q(x)},$$

where p and q are the polynomials defined by (5.17) and (5.18), respectively. A simple computation shows that $S(x) \Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(x) = \sigma_{x-\frac{m-1}{2}} P(x)$.

Notice that the rational function S is the particular case of the rational function (5.23) in Lemma 5.2 for $\Xi = 1$. Since obviously $\Xi = 1$ is a polynomial invariant under the action of $\mathcal{I}^{a+b-m-1}$, we get, as a direct consequence of Lemma 5.2 and Theorem 3.1, that the polynomials q_n , $0 \leq n \leq N + m_3 + m_4$, are eigenfunctions of a higher-order difference operator $D_{q,S}$ in the algebra of difference operators \mathcal{A} (3.1), explicitly given by (3.11).

We now compute the order of $D_{q,S}$.

Since $S\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}} = \sigma_{x-\frac{m-1}{2}}P$, Lemma 5.1 gives that the degree of $S\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}$ is $d+1$, where d is the degree of P given by

$$d = 2 \sum_{u \in U_1, U_2, U_3, U_4} u - 2 \sum_{i=1}^4 \binom{m_i}{2},$$

(notice that the assumption (5.19) in Lemma 5.1 is just (6.7) above). Hence the polynomial P_S defined by

$$P_S(\theta_x) - P_S(\theta_{x-1}) = S(x)\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(x) + S(x+m)\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(x+m)$$

has degree $d/2 + 1$ (since θ_x is a polynomial of degree 2), that is,

$$\sum_{u \in U_1, U_2, U_3, U_4} u - \sum_{i=1}^4 \binom{m_i}{2} + 1.$$

Taking into account that the m -tuple G (6.6) is formed by the sets $J_{h_i}(F_i)$, $i = 1, 2, 3$, and $I(F_4)$, the definitions of the involution I (4.8) and the transform J_h (4.9) give

$$\begin{aligned} & \sum_{u \in U_1, U_2, U_3, U_4} u - \sum_{i=1}^4 \binom{m_i}{2} + 1 \\ &= \sum_{f \in F_4} f - \sum_{f \in F_1, F_2, F_3} f - \sum_{i=1}^4 \binom{n_{F_i}}{2} + \sum_{i=1}^3 n_{F_i}(f_{i,M} + h_i) + 1 \\ &= r. \end{aligned}$$

That is, P_S is a polynomial of degree r . Consider the coefficients B and D of \mathfrak{s}_1 and \mathfrak{s}_{-1} in the second-order difference operator $D_{a,b,N}$ for the Hahn polynomials (5.3). We then deduce that the operator $\frac{1}{2}P_S(D_{a,b,N})$ has the form

$$\sum_{l=-r}^r \tilde{h}_l(x) \mathfrak{s}_l,$$

where $\tilde{h}_r(x) = u_S \prod_{j=0}^{r-1} B(x+j)$, $\tilde{h}_{-r}(x) = u_S \prod_{j=0}^{r-1} D(x-j)$ and u_S denotes the leading coefficient of the polynomial P_S . Using (5.3), we deduce that both \tilde{h}_{-r} and \tilde{h}_r are polynomials of degree $2r$.

Consider now the coefficients \tilde{B}_h and \tilde{D}_h of Δ and ∇ in any of the \mathcal{D} -operators \mathcal{D}_h for the Hahn polynomials (see (5.11) and (5.12)). Notice that $\tilde{B}_h = 0$, $h \in \mathbb{U}_1 \cup \mathbb{U}_4$, and $\tilde{D}_h = 0$, $h \in \mathbb{U}_2 \cup \mathbb{U}_3$. Using Lemmas 3.4 and 5.1, we can conclude that the polynomials M_h (3.7) have degree at most $v_h = 2r - 2g_h - 1$. This implies that the polynomials \tilde{M}_h (3.9) have degree at most $r - g_h - 1$.

Assume now $h \in \mathbb{U}_1 \cup \mathbb{U}_4$. Since Y_h has degree g_h , we get that the operator $\tilde{M}_h(D_{a,b,N})\mathcal{D}_h Y_h(D_{a,b,N})$ has the form

$$\sum_{l=-r}^{r-1} \tilde{h}_l(x) \mathfrak{s}_l,$$

where

$$\tilde{h}_{-r}(x) = u_Y u_M \tilde{D}_h(x - v_h) \prod_{j=0; j \neq v_h}^{r-1} D(x - j),$$

u_Y is the leading coefficient of Y_h and u_M is the coefficient of x^{v_h} in \tilde{M}_h . As before, we deduce that \tilde{h}_{-r} is a polynomial of degree $2r - 1$.

On the other hand, if $h \in \mathbb{U}_2 \cup \mathbb{U}_3$, since Y_h has degree g_h , we get that the operator $\tilde{M}_h(D_{a,b,N})\mathcal{D}_h Y_h(D_{a,b,N})$ has the form

$$\sum_{l=-r+1}^r \tilde{h}_l(x) \mathfrak{s}_l,$$

where

$$\tilde{h}_r(x) = u_Y u_M \tilde{B}_h(x - v_h) \prod_{j=0; j \neq v_h}^{r-1} B(x - j).$$

As before, we deduce that \tilde{h}_r is a polynomial of degree $2r - 1$.

To complete the proof of (2) it is enough to take into account the expression of $D_{q,S}$ given by (3.11). □

Corollary 6.3. *Let $\mathcal{F} = (F_1, F_2, F_3, F_4)$ be a quartet of finite sets of positive integers (the empty set is allowed, in which case we take $\max F = -1$). Let a and b be real numbers satisfying*

$$\begin{aligned} a, b, a + b &\neq -1, -2, -3, \dots, \\ a + f_{2,M} + f_{4,M} + 1 &\neq 0, 1, 2, \dots, \quad \text{if } F_2 \text{ or } F_4 \neq \emptyset, \quad \text{and} \\ b + f_{1,M} + f_{3,M} + 1 &\neq 0, 1, 2, \dots, \quad \text{if } F_1 \text{ or } F_3 \neq \emptyset. \end{aligned}$$

Consider the weight $\rho_{a,b,N}^{\mathcal{F}}$ defined by

$$(6.8) \quad \rho_{a,b,N}^{\mathcal{F}} = \prod_{f \in F_1} (b + N + 1 + f - x) \prod_{f \in F_2} (x + a + 1 + f) \prod_{f \in F_3} (N - f - x) \prod_{f \in F_4} (x - f) \rho_{a,b,N},$$

where $\rho_{a,b,N}$ is the Hahn weight (5.5). Assume that

$$\Omega_{a+f_{2,M}+f_{4,M}+2, b+f_{1,M}+f_{3,M}+2, N-f_{3,M}-f_{4,M}-2}^{\mathcal{U}}(n) \neq 0, \quad 0 \leq n \leq N + m_3 + m_4 + 1,$$

where $\mathcal{U} = (I(F_1), I(F_2), I(F_3), I(F_4))$. Then the measure $\rho_{a,b,N}^{\mathcal{F}}$ has associated a sequence of orthogonal polynomials and they are eigenfunctions of a higher-order difference operator of the form (1.1) with

$$(6.9) \quad -s = r = \sum_{f \in F_1, F_2, F_3, F_4} f - \sum_{i=1}^4 \binom{n_{F_i}}{2} + 1$$

(which can be constructed using Theorem 3.1).

Proof. If we write $\tilde{F}_j = \{f_{j,M} - f + 1, f \in F_j\}$, $j = 1, 2, 3$, and $h_j = \begin{cases} \min F_j, & \text{if } F_j \neq \emptyset, \\ 1, & \text{if } F_j = \emptyset, \end{cases}$

using (4.8) and (4.9), one straightforwardly has $J_{h_j}(\tilde{F}_j) = I(F_j)$. Write now $\tilde{\mathcal{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, F_4)$. It is now easy to see that

$$\rho_{a,b,N}^{\mathcal{F}} = \tilde{\rho}_{a+f_{2,M}+f_{4,M}+2, b+f_{1,M}+f_{3,M}+2, N-f_{3,M}-f_{4,M}-2}^{\tilde{\mathcal{F}}, \mathfrak{h}}(x - f_{4,M} - 1).$$

The corollary is then a straightforward consequence of the previous Theorem. □

Remark 6.4. The hypothesis on $\Omega_{a,b,N}^{\mathcal{U},\mathfrak{h}}(n) \neq 0$, for $0 \leq n \leq N + m_3 + m_4 + 1$, in the previous Theorem and Corollary is then sufficient for the existence of a sequence of orthogonal polynomials with respect to the (possible signed) measure $\tilde{\rho}_{a,b,N}^{\mathcal{F},\mathfrak{h}}$. We guess that this hypothesis is also necessary for the existence of such sequence of orthogonal polynomials.

Remark 6.5. Notice that there are different sets F_3 and F_4 for which the measures $\rho_{a,b,N}^{\mathcal{F}}$ (6.8) are equal. Each of these possibilities provides a different representation for the orthogonal polynomials with respect to $\rho_{a,b,N}^{\mathcal{F}}$ in the form (6.4) and a different higher-order difference operator with respect to which they are eigenfunctions. It is not difficult to see that only one of these possibilities satisfies the condition $f_{3,M}, f_{4,M} < N/2$. This is the more interesting choice because it minimizes the order $2r$ of the associated higher-order difference operator. This fact will be clear with an example. Take $N = 100$ and the measure $\mu = (x-1)(x-5)(x-68)\rho_{a,b,N}^{\mathcal{F}}$. There are eight couples of different sets F_3 and F_4 for which the measures μ and $\rho_{a,b,N}^{\mathcal{F}}$ (6.8) coincide (except for a sign). They are the following

$$\begin{aligned} F_3 = \emptyset, F_4 = \{1, 5, 68\}, & \quad F_3 = \{32, 95, 99\}, F_4 = \emptyset, \\ F_3 = \{32\}, F_4 = \{1, 5\}, & \quad F_3 = \{95\}, F_4 = \{1, 68\}, & \quad F_3 = \{99\}, F_4 = \{5, 68\}, \\ F_3 = \{32, 95\}, F_4 = \{1\}, & \quad F_3 = \{32, 99\}, F_4 = \{5\}, & \quad F_3 = \{95, 99\}, F_4 = \{68\}. \end{aligned}$$

Only one of these couples satisfies the assumption $f_{3,M}, f_{4,M} < N/2$: $F_3 = \{32\}$, $F_4 = \{1, 5\}$. Actually, it is easy to check that this couple minimizes the number

$$\sum_{f \in F_3, F_4} f - \binom{n_{F_3}}{2} - \binom{n_{F_4}}{2} + 1.$$

Hence, it also minimizes the order $2r$ (see (6.9)) of the difference operator with respect to which the polynomials $(q_n)_n$ are eigenfunctions.

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